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THE DIRICHLET PROBLEM FOR THE TIME–FRACTIONAL HEAT CONDUCTION EQUATION WITH HEAT ABSORPTION IN A MEDIUM WITH SPHERICAL CAVITY

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The subject of consideration is the time–fractional heat conduction equation with one spatial variable in spherical coordinate system in a medium with spherical cavity. The fundamental solution to the Dirichlet problem is obtained using the integral transform technique. The numerical results are illustrated grafically.

1. The Dirichlet problem

The time–fractional heat conduction equation

$$\frac{\partial^\alpha T(r,t)}{\partial t^\alpha}=a\left(\frac{\partial^2 T(r,t)}{\partial r^2}+\frac{2}{r}\frac{\partial T(r,t)}{\partial r}\right)-bT(r,t)$$

$$0<\alpha\le 2,\ R<r<\infty,\ 0<t<\infty,\ a>0$$

$$\frac{d^\alpha T}{d t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n T(\tau)}{d\tau^n}\mathrm{d}\tau,\quad n-1<\alpha<n$$

The boundary condition

$$T(R,t)=p_0\,\delta(t)$$

Zero initial conditions

$$T(r,0)=0,\qquad 0<\alpha\le 2$$

$$\frac{\partial T(r,0)}{\partial t}=0,\qquad 1<\alpha\le 2$$

2. The initial–boundary–value problem for the auxiliary variable and the new sought–for function

$$x=r-R,\qquad u=rT$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=a\frac{\partial^2 u(x,t)}{\partial x^2}-bu(x,t),\qquad 0<x<\infty$$

$$u(0,t)=R\,p_0\,\delta(t)$$

$$u(x,0)=0,\qquad 0<\alpha\le 2$$

$$\frac{\partial u(x,0)}{\partial t}=0,\qquad 1<\alpha\le 2$$

3. The fundamental solution

Application of the sin-Fourier transform with respect to the spatial coordinate x and the Laplace transform with respect to the time t

$$\tilde{u}^*(\xi,s)=\frac{aR\,p_0\xi}{s^\alpha+a\xi^2+b}$$

Inversion of the integral transforms

$$u(x,t)=\frac{2a\,p_0R}{\pi}t^{\alpha-1}\int_0^\infty E_{\alpha,\alpha}\left[-\left(a\xi^2+b\right)t^\alpha\right]\xi\sin(\xi x)\,\mathrm{d}\xi$$

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha,\beta>0$$

The fundamental solution

$$T(r,t)=\frac{2a\,p_0R}{\pi r}t^{\alpha-1}\int_0^\infty E_{\alpha,\alpha}\left[-\left(a\xi^2+b\right)t^\alpha\right]\xi\sin[(r-R)\xi]\,\mathrm{d}\xi$$

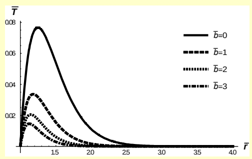
4. The nondimensional quantities for numerical calculations

$$\bar r=\frac{r}{R},\qquad \bar\xi=R\xi,\qquad \kappa=\frac{\sqrt{a}\,t^{\alpha/2}}{R},\qquad \bar b=bt^\alpha,\qquad \bar T=\frac{t}{p_0}T$$

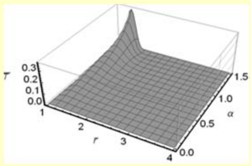
The solution

$$\bar T=\frac{2\kappa^2}{\pi\bar r}\int_0^\infty \bar\xi\sin[(\bar r-1)\bar\xi]E_{\alpha,\alpha}\left(-\kappa^2\bar\xi^2-\bar b\right)\mathrm{d}\bar\xi$$

5. The results of numerical calculations



The fundamental solution to the Dirichlet problem for $\alpha=0{,}5,\ \kappa=0{,}25$.



The fundamental solution to the Dirichlet problem for $0\le\alpha\le 1{,}5$ ($\kappa=0{,}25,\ \bar b=0{,}5$).

The classical theory of heat conduction predicts that the effects of a thermal disturbance will be felt instantaneously at distances infinitely far from its source. This limitation of the theory follows from the fact that the classical heat conduction equation is a parabolic-type equation. The hyperbolic-type heat transport equation allows finite wave speed for thermal signals. This phenomenon is known as second sound. In the case of time fractional heat conduction equation with $1<\alpha<2$ the propagating peaks approximating delta function are also exhibited. The points, where the fundamental solution takes its maximum, propagate with finite speed.

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